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MATHEMATICAL MONTHLY.

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BIOGRAPHY.

ELISHA SCOTT LOOMIS, A. M., PH. D.

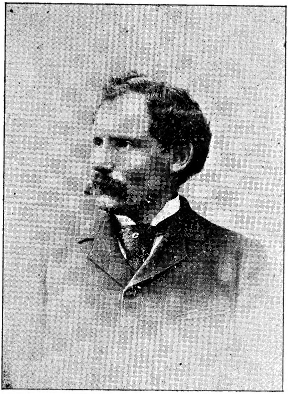
BY B. F. FINKEL.

PROFESSOR LOOMIS was born near the village of Wadsworth, Medina county, Ohio, September 18th, 1852, and is the oldest of a family of eight children, seven boys and one girl. He is of English-Scotch and Pennsylvania Dutch descent.

The Loomises came to Portage county, Ohio, from Southampton, Mass., about 1788, but originally the Loomis family dwelt at Windsor, Connecticut, where one Joseph Loomis settled in 1639, coming from Braintree, England.

Professor Loomis being left fatherless at the age of twelve and his widowed mother being left homeless and penniless, he was obliged to help care for his mother and two younger brothers, the babies, as best he could. At the age of thirteen years six months, he hired out to a farmer for six months at \$3.00 per month, waiting nearly a year before he received his wages. In the spring of 1867, he hired out for eight months at \$5.00 per month. Out of this hard earned sum, he loaned \$25.00 at 6%, being his first earnings that he did not give to his mother for her support. Working for the same man for the following three years, his wages rose to \$6¼, \$8, and \$10, per month. The next two years we find him working on another farm at \$13.00 and \$16.00 per month. In this school of adversity, he learned what many young men ever fail to learn, he learned the lessons of patience, economy, frugality, and industry. Through all these years of toil, he succeeded in attending the district school about four months each winter, working for his board while doing so.

While a pupil in a small district school in the hills of Holmes county, Ohio, he desired to learn the beauty and mystery of Algebra. Having walked seven



Elisha Scott Loomis, A. M., Ph. D.

miles to a neighboring town to procure a book, he purchased Ray's Elementary Algebra, without a teacher (the district school teacher knew nothing about Algebra) took up the study, conquered the difficulties, and reached equations of the first degree in the short time of two months. This he did without a word of help or encouragement from any one except his mother who encouraged him as best she could, for she never attended school herself but a few terms previous to her twelfth year, and was never permitted to learn to write.

Possessing, however, a thirst for learning and being determined to go to college (this he inherited from his grandmother Loomis as he believes, she being a descendant from an educated Scotch family), he taught school the summer 1873, teaching 54 days for $50, and boarding himself. His savings now amounted to $250. In the fall of 1873, he went to Baldwin University, Berea, Ohio, and commenced a college course beginning at the foot of the ladder. Teaching during the winters, applying himself very diligently to his books in private and working in the harvest fields during summer vacations, at the same time doing his part in caring for his mother, remaining out of college long enough to earn money by teaching at $60 per month, to buy a home in Shreve, Ohio, into which he moved his mother and brothers the fall of 1876, he finally graduated, June 10, 1880.

On the 17th of June, 1880, he married Miss Letitia E. Shire, who was engaged in teaching near Loudonville, Ohio. Taking charge of the Burbank Academy and public schools of the same place, they found time to take up the course of reading in the C. L. S. C., which they finished together, graduating in the same class in 1884, at Chautauqua, New York.

In 1881, Professor Loomis was asked to take charge of the Richfield High School in Summit county, Ohio, which position he accepted and filled for four years to the entire satisfaction of his patrons. In the summer of 1885, he went to south Kansas and while there the trustees of Baldwin University, Berea, Ohio, sought him out, and in August elected him to the chair of Mathematics in that Institution. This position Professor Loomis now holds with honor to himself and the University. His predecessor was Dr. Aaron Schuyler, the very able teacher and author of a series of excellent mathematical text-books.

While Professor Loomis has a special fondness for mathematical studies, yet his devotion to them did not prevent him from investigating other special subjects; and that he might be the better teacher and broader thinker, he took the post-graduate course on metaphysics and social science in the University of Wooster, receiving, in recognition of his sound scholarship, the degree of Ph. D., June 20, 1888. The subject of the thesis submitted for the above degree was "Theism the Result of Completed Investigation." This thesis exhibits much thought and wide reading on the part of the author.

Dr. Loomis has given instruction to correspondence students in eight different states. Among them was a Professor in a western college who took a course in Williamson's Differential Calculus. He has been a contributor to some of the leading Mathematical Journals of the United States and is at present a valued contributor to the MONTHLY. Dr. Loomis also holds a High-School

Life Certificate issued by the State of Ohio; it covers 21 branches and is signed by some of the ablest educational men of Ohio. He says he prizes it more highly than he does his first diploma, because it means more, educationally.

Having taught in the various grades of schools from the primary district school to the college, Dr. Loomis has made a careful study of the methods of teaching, having carefully read some fifty different pedagogical works, with a view to discover the true Psychic law of how one soul communicates to another most economically and clearly any piece of knowledge. As a result, he has learned to clothe the "dry bones of Mathematics" so that his students say, "this is not dry, this is interesting."

Needing more outdoor exercise than he was wont to take, Prof. Loomis took up the subject of engineering. He is now the engineer for the village of Berea, as well as a member of the Society of Civil Engineers of Ohio.

Having made a public confession of Christ when but nineteen years of age, uniting first with the Presbyterian church and afterwards with the Methodist, Dr. Loomis wishes to say for the encouragement of young men just entering upon life's duties that the best counsel and guide any young man can choose is the ten commandments and the Lord's Prayer. He who lives up to the full measure found therein will be sought out and given posts of duty and honor. Honor thy father and thy mother, love thy neighbor as thyself, and forgive thine enemy, are factors which cannot be omitted in solving the great problems of life here and hereafter.

Dr. Loomis can say what any young man ought to be proud to say, but what too few are able to say: viz., that he has never purchased or used a cent's worth of tobacco in his life, and that he has never stood before the bar of a drinking saloon and asked for, or drank that which comes from over the bar of such a place, always having been a pronounced opponent of tobacco and intoxicating drinks in all their forms.

To day, many young men who are failing in life, beholding those who are succeeding, say "why have luscious plumbs of success fallen in their mouths while in mine have fallen the bitter almonds of adversity. But the answer is: Plumbs do not fall in a successful man's mouth by chance; he takes hold of the tree upon which fortune hangs and shakes it bravely and manfully. Providence does not shower fortune and fame on some and poverty and degradation on others. Blasphemeus is he who would attribute to the loving Father all the misery, wretchedness, and woe that are in the world, when they are the results of violated law, outraged conscience. He who would rise to positions of honor and distinction must do so by honest persistent effort. Dr. Loomis had heartaches and discouragements which would have conquered a less resolute spirit but he was not to be vanquished in the struggle for the comsummation of his noble purposes. How consoling it is for those who think the world is against them to read of the struggles and disappointments, the trials and temptations of those who have risen from penury and want, from the humble and despised walks of life to positions of honor through their effort. Thus was Garfield: day before yesterday on the toe-path, yesterday president of the United States,

and to-day a hero of eternity. Thus it is with thousands of men all over the world to-day. When they tread the wine press alone, they looked to higher callings. But they not only looked to higher callings, they worked for higher things and putting their trust in their strong right arm and the power of God, they batterd down the bulwarks between them and the world of fortune and success. While the slumberers slept they planned for the morrow, for the week, for the month, for the year, yea for life here and hereafter. This is true of Dr. Loomis. Possessing high aims and a strong will he is bound to succeed in whatever he undertakes,obstacles having been and are only incentives to more vigorous action. Building on the foundation of temperance, honesty, and industry, Dr. Loomis will succeed in whatever field he may be called, his motto being *Viam Inviniam aut Faciam.*

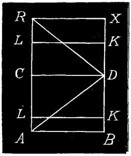
NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D , (Johns Hopkins): Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

CHAPTER SECOND.

[Continued from the June Number.]

PROPOSITION V. *The hypothesis of right angle, if even in a single case it is true, always in every case it alone is true.*



PROOF. Let the join CD (Fig. 4) make right angles with any two equal perpendiculars AC, BD, standing upon any other AB.

CD will be (P. III.) equal to this AB. Assume in AC, and BD produced two sects CR, DX, equal to these AC, BD; and join RX. We may easily show that the join RX will be equal to this AB, and the angles at it right. And first indeed by superposition of the quadrilateral $ABDC$ upon the quadrilateral $CDXR$, applied to the common base CD.

More elegantly then we may proceed thus. Join AD, RD. It follows (Eu. I. 4.) in the triangles ACD, RCD, the bases AD, RD will be equal and likewise the angles CDA, CDR, and certainly ADB, RDX, because equal remainders from a right angle.

Whereby in turn (Eu. I. 4.) in the triangles ADB, RDX, the base AB will be equal to the base RX. Therefore (from the preceding proposition)

the angles at the join RX will be right, and therefore we abide in the same hypothesis of right angle.

Since however the length of the perpendiculars can be similarly increased indefinitely, under the same base AB, the hypothesis of right angle subsisting, it is to be demonstrated that the same hypothesis will always abide in any case of diminution of those perpendiculars; which indeed is thus evinced.

Assume in AR, and BX any two equal perpendiculars AL, BK and join LK. If the angles at the join LK are not right, nevertheless they will be (P. I.) equal to each other.

Therefore they will be toward the one part, as suppose toward AB obtuse, and toward RX acute, since certainly the angles here at each of those points are equal (Eu. I. 13.) to two rights. But it also holds that the perpendiculars LR, KX, those standing on RX, will be mutually equal. Therefore (P. III.) LK will be greater indeed than the opposite RX, and less than the opposite AB.

But this is absurd; because AB, and RX have been shown equal.

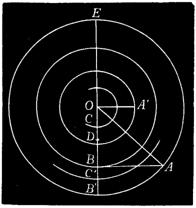
Therefore the hypothesis of right angle is not changed by any diminution of the perpendiculars, whilst abides the once posited base AB.

But neither is the hypothesis of right anglechanged for any diminution, or greater amplitude of the base; since manifestly may be considered as any base the perpendicular BK, or BX, and therefore may be considered in turn as perpendiculars that AB, and the equal opposite sect KL, or XR. Therefore is established that the hypothesis of right angle, if even in a single case it be true, always in every case it alone is true. Quod erat demonstrandum.

A PROOF OF THE PHYTHAGOREAN PROPOSITION.

By Professor ANDREW INGRAHAM, President of the Swain Free School, New Bedford, Massachusetts.

I. The circle whose radius is the hypothenuse is equivalent to the two circles having the base and the altitude for their respective radii.



Let the triangle OBA, rightangled at B, revolve in the plane of the paper about O. The diagram shows that the circle generated by OA is equivalent to the circle generated by OB and the annule generated by BA.

This proves the proposition to one who knows that the annule generated by BA is equivalent to the circle whose radius is BA.

II. The annule generated by a straight line perpendicular to a given line from any point in the same, while the latter revolves in their common plane around any

of its own points as a centre, is equivalent to the circle which has the former line for radius.

Let OC be named R. Take $OD=2R$, $OC'=lR$, $BC'=C'B'=\frac{R}{l}$.

Now $BA=\sqrt{EB.\ BB'}=\sqrt{\left(lR+\frac{R}{l}+lR-\frac{R}{l}\right)\frac{2R}{l}}=2R=OA'$ and

$$OC:\ OC'=BC':\ OC=BB':\ OD.$$

Circum. OC : Circum. $OC'=BB'$: OD.

$\therefore$ OD. Circum. $OC=BB'$.Circum. OC'. Circle OA'=annule BA.

THE NEBULAR HYPOTHESIS.

By REV. A. L. GRIDLEY, Pastor of Congregational Church, Kidder, Missouri.

About twenty-five years ago while the writer of this atricle was a student in college, in speaking of the Nebular Hypothesis he told the Professor of mathematics that he believed that the impossibility of the said hypothesis could be proved by mathematics. He believed that the contraction could not have proceeded in such a way as to produce the necessary velocities at various times to perpetuate a revolution around the center.

With this belief he began work but had not proceeded far before he found that so far from disproving the possibility of the hypothesis he had discovered a clue by which he could estimate within very narrow limits the necessary rate of contraction from the orbit of one planet to that of another.

He would now venture nothing as to the truth of the hypothesis but if it be a true one,and the Solar Systen has evolved from a former nebulous mass, the time necessarily occupied in forming the liquid globes now constituting the planets of our system can be closely calculated.

In compliance with repeated requests from one of the editors of the "Journal" the calculations are here given.

The law may be briefly stated as follows: *The orbital velocity of any interior planet is the resultant of the orbital velocity of the next exterior planet and the contracting force.* For example the orbital velocity of Jupiter is greater than that of its exterior planet Saturn. Its velocity is the resultant of the velocity of rotation when the mass reached to Saturn and of the contracting force from that point to the present orbit of Jupiter. We have then the resultant of two forces and one of the forces to find the other and any one familiar with mechanics would give the rule: *From the square of the resultant subtract the square of the known force and the square root of the remainder will be the required force.*

Applying this rule we ascertain the rate of contraction from one orbit to another. Let N= orbital velocity of Neptune, U= that of Uranus, D= the distance between the orbits, T= the time of contraction from one orbit to the other. Then $T=\frac{D}{(U^2-N^2)}$.

Substituting for D, U and N, their known values and T in this case =4156 sidereal terrestrial days.

By the same process we find the time from Uranus to Saturn to be 244 such days, from Saturn to Jupiter to be 85 days. Passing to the minor planets, the time from Mars to the earth =26 days, Earth to Venus =26.4 days Venus to Mercury 17.7 days.

Of course these computations are confined to the conditions after the nebula had contracted to the limits of the present system.

By comparing the velocity of Neptune with the velocity of a body falling from infinite space probably a limit could be set beyond which the nebula could not have extended. But the original velocity of rotation cannot be ascertained by any known circumstance although it would be rash to speak positively even upon that point.

Minor circumstances not here taken into consideration might to a limited extent, affect the results just given, but upon the whole, the law would probably be subject to as little variation as Kepler's Third Law.

The radiation of heat from the mass would have been so rapid that contraction could not have proceeded steadily from Neptune to the sun and the mass broke into two pieces where the asteroids now are and these small planets are the "splinters" resulting from the separation. Thus some of the interior planets and Jupiter were forming contemporaneously.

The velocity of axial rotation of the planets depends upon the thickness of the rings deposited and of course size of the planet depends upon the same circumstance and so the larger the planet the greater the rate of axial rotation. This circumstance accounts for "Kirkwood's Law."

From the velocity of axial rotation and the distance of the satellites from the primary planets the minimum thickness, at least, of the rings can be approximately ascertained.

By following out the clues here given many things may be learned more curious than useful perhaps,but interesting from a scientific standpoint,if we admit the truth of the hypothesis.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

19. Proposed by MISS LECTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder, Missouri.

Bought sugar at $6\frac{1}{2}$ cents a pound: waste by transportation and retailing was 5%; interest on first cost to time of sale was 2%. How much must be asked per pound to gain 25%?

Solution by H. R. YOUNG, West Sunbury, Pa. and A. L. FOOTE, C. E., No. 80 Broad St. New Yerk City.

For 100 pounds he must pay $6.50.

Interest at 2% = $.13 makes $6.63, the cost.

125% of $6.63 = $$8.28\frac{3}{4}$, what he must charge.

$\$8.28\frac{3}{4}\div(100\times.86)=.08\frac{55}{76}=8\frac{55}{76}$ cents per pound.

Also sclved by *I. L. BEVERAGE, P. C. CULLEN, J. K. ELLWOOD, P. S. BERG,* and *G. B. M. ZERR.*

20. Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.

I owed a merchant $600. The merchant agreed to take part of the amount and wait a year for the balance, if I would pay interest in advance. I paid $300. How much of this was interest on the unpaid balance, and how much went toward the payment of debt?

I. Solution by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Pennsylvania, and Professor G. B. M. ZERR, Principal of High School, Staunton, Virginia.

Let P=payment. Then $(600-P).06+P=300$. $\therefore .94P=264$,

$\therefore$ $P=\$280.85\frac{5}{47}$. $\$300-\$280.85\frac{5}{47}=\$19.42\frac{42}{47}$, interert on unpaid debt.

II. Solution by I. L. BEVERAGE, Monterey, Virginia, and JOHN T. FAIRCHILD, Ada, Ohio.

Let 100% be the face of the new note. Then 6% is the interest, and ($600−100%) is the amount paid on the debt. But, ($300−6%) = amount paid on the debt.

$\therefore$ ($600−100%) = ($300−6%), or 94% = $300.

$\therefore$ 1% = $3.1915. $\therefore$ 100% = $319.15, new note.

$\therefore$ $319.15 × .06 = $19.15, interest on the unpaid balance, and $600−$319.15 = $280.85, amount paid on the debt.

This problem was also solved by *P. S. BERG, R. H. YOUNG, A. L. FOOTE, P. C. CULLEN,* and *JOHN FAUGHT.*

21. Proposed by A. L. FOOTE, C. E., No. 80, Broad St., New York City.

A merchant bought a certain quantity of corn for which he paid a certain sum of money; but on measuring he found only $\frac{39}{40}$ of quantity he expected. He sold it gaining $\frac{1}{8}$ of the cost and received $2,160, which was at the rate of $12\frac{4}{13}$ cents per bushel more than he would have paid had he received the quantity expected. How many bushels did he suppose he had bought, and at what price?

[Selected from *Robinson's Arithmetical Problems.*]

Solution by JOHN FAUGHT, Vincennes University, Vincennes, Indiana.

1. $\frac{1}{8}$ of cost + cost = $\frac{9}{8}$ of cost = \$2160.
2. $\therefore$ the cost = \$1920.
3. The number of bushels $\times$ cost per bu. = \$1920.
4. $\frac{39}{40}$ number of bu. $\times$ cost per bu. = $\frac{39}{40}$ of \$1920 = \$1872.
5. $\frac{39}{40}$ of number of bu. $\times$ cost per bu. + $\frac{39}{40}$ of number of bu. $\times \$0.12\frac{4}{13}$ = \$2160.
6. $\therefore$ \$1872 + $\frac{39}{40}$ of number of bu. $\times \$0.12\frac{4}{13}$ = \$2160,
7. or $\frac{39}{40}\times\frac{1.60}{13}$ of number of bu. $=2160-1872=288$,
8. or $.12\times$ number of bu. $=288$.
9. $\therefore$ number of bu. $=2400$.
10. $\$1920\div2400=\$0.80$, price per bushel.

$\therefore$ he bought 2400 bushels at 80 cents per bushel.

This problem was also solved by *G. B. M. ZERR*, *R. H. YOUNG*. *A. L. FOOTE*, *COOPER D. SCHMITT*, *P. S. BERG*. *P. C. CULLEN*, *I. L. BEVERAGE*, *J. K. ELLWOOD*, and *W. I. TAYLOR*.

PROBLEMS

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A and B buy a ship for $S=\$80000$, of which A has the abth $=\frac{5}{8}$, and B the cdth $=\frac{3}{8}$, interest. They sell C the mnth $=\frac{1}{3}$ interest for $P=\$40000$; and then agree that A should *retain* the pqth $=\frac{7}{12}$, and B the rsth $=\frac{1}{12}$, interest. How is the purchase-money received from C to be divided between A and B?

28. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A rectangular field (not square) contains as many acres as there are boards in the fence enclosing it. The fence is 4 boards high and each board is 11 feet long. How many acres in the field?

Solutions to these problems should be received on or before September 1st.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

19. Proposed by A. L. FOOTE, C. E., No. 80, Broad St., New York City.

$$\text{Given}\left\{\begin{array}{c}\dfrac{xyz}{\sqrt[n]{(x^n+y^n)}}\\[2mm] \dfrac{xyz}{\sqrt[n]{(x^n+z^n)}}\\[2mm] \dfrac{xyz}{\sqrt[n]{(y^n+z^n)}}\end{array}\right\}\quad\text{To find } x,\ y, \text{ and } z.$$

I. Solution by B. F. FINKEL, A.M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

From the equations, we have

$$\frac{x^ny^nz^n}{x^n+y^n}=a^n\ldots.(1);\ \frac{x^ny^nz^n}{x^n+z^n}=b^n\ldots.(2);\ \text{and}\ \frac{x^ny^nz^n}{y^n+z^n}=c^n\ldots.(3).$$

Let $x^ny^nz^n=p$ and $x^n+y^n+z^n=s$. Then from (1), we have $\dfrac{p}{s-z^n}=a^n\ldots.(4)$; from (2), $\dfrac{p}{s-y^n}=b^n\ldots.(5)$; from (3), $\dfrac{p}{s-x^n}=c^n\ldots.(6)$.

From (4), (5), and (6) we get $z^n=s-\dfrac{p}{a^n}\ldots.(7)$, $y^n=s-\dfrac{p}{b^n}\ldots.(8)$, and $x^n=s-\dfrac{p}{a^n}\ldots.(9)$, respectively. Adding (7), (8), and (9), we have $x^n+y^n+z^n=3s-p\left(\dfrac{1}{a^n}+\dfrac{1}{b^n}+\dfrac{1}{c^n}\right)$. But $x^n+y^n+z^n=s$, hence $s=3s-p\left(\dfrac{1}{a^n}+\dfrac{1}{b^n}+\dfrac{1}{c^n}\right)\ldots.(10)$. Whence $s=\dfrac{[(ab)^n+(ac)^n+(bc)^n]p}{2(abc)^n}\ldots.(11)$.

Substituting the value of s in (7), (8), and (9), we have

$$z^n=\frac{[(ab)^n+(ac)^n-(bc)^n]p}{2(abc)^n}\ldots.(12);\quad y^n=\frac{[(ab)^n-(ac)^n+(bc)^n]p}{2(abc)^n}\ldots.(13);$$

$x^n=\dfrac{[(ac)^n-(ab)^n+(bc)^n]p}{2(abc)^n}\ldots.(14)$. Multiplying (7), (8), and (9) together, we have $x^ny^nz^n$,

$$\text{or } p=\frac{[(ab)^n+(ac)^n-(bc)^n][(ac)^n-(ab)^n+(bc)^n][(bc)^n-(ac)^n+(ab)^n]p^3}{8(abc)^{3n}}$$

$$\text{whence, } p=2(abc)^n\sqrt{\frac{2(abc)^n}{[(ab)^n+(ac)^n-(bc)^n][(ac)^n-(ab)^n+(bc)^n][(bc)^n-(ac)^n+(ab)^n]}}$$

Substituting this value of p in (12), (13), and (14) and extract the nth root of the resulting equation, we have

$$x=\sqrt[n]{\left\{\frac{2(abc)^n[(ac)^n+(bc)^n-(ab)^n]}{[(ab)^n+(ac)^n-(bc)^n][(bc)^n-(ac)^n+(ab)^n]}\right\}},$$

$$y=\sqrt[n]{\left\{\frac{2(abc)^n[(ab)^n-(ac)^n+(bc)^n]}{[(ab)^n+(ac)^n-(bc)^n][(ac)^n-(ab)^n+(bc)^n]}\right\}},$$

$$z=\sqrt[n]{\left\{\frac{2(abc)^n[(ab)^n+(ac)^n-(bc)^n]}{[(ab)^n+(bc)^n-(ac)^n][(ac)^n+(bc)^n-(ab)^n]}\right\}}.$$

ERRATUM.—In last line on preceding page, for "equation" read *equations.*

II. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Raising each equation to the nth power we have at once

$$a^n(x^n+y^n)=b^n(x^n+z^n)=c^n(y^n+z^n)=(xyz)^n.$$

Dividing these severally by $(xyz)^n$ we have the 3 equations,

$$\frac{1}{(yz)^n}+\frac{1}{(xz)^n}=\frac{1}{a^n};\ \frac{1}{(yz)^n}+\frac{1}{(xy)^n}=\frac{1}{b^n};\ \text{and}\ \frac{1}{(xz)^n}+\frac{1}{(xy)^n}=\frac{1}{c^n}.$$

Adding these three and dividing by 2, we have

$$\frac{1}{(xy)^n}+\frac{1}{(xz)^n}+\frac{1}{(yz)^n}=\frac{1}{2}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right);s=$$ then from this result, subtract each of the 3 equations seperately, and we get

$$\frac{1}{(xy)^n}=s-\frac{1}{a^n},\ \frac{1}{(xz)^n}=s-\frac{1}{b^n},\ \frac{1}{(yz)^n}=s-\frac{1}{c^n}.$$

Multiplying these three we have

$$\frac{1}{(xyz)^{2n}}=\left(s-\frac{1}{a^n}\right)\left(s-\frac{1}{b^n}\right)\left(s-\frac{1}{c^n}\right),$$

or $(xyz)^{2n}=\dfrac{1}{\left(s-\frac{1}{a^n}\right)\left(s-\frac{1}{b^n}\right)\left(s-\frac{1}{c^n}\right)}$.

$$\text{Whence } \frac{(xyz)^{2n}}{(yz)^{2n}}=x^{2n}=\frac{\dfrac{1}{\left(s-\frac{1}{a^n}\right)\left(s-\frac{1}{b^n}\right)\left(s-\frac{1}{c^n}\right)}}{\left(s-\frac{1}{c^n}\right)^2}=\frac{s-\frac{1}{c^n}}{\left(s-\frac{1}{a^n}\right)\left(s-\frac{1}{b^n}\right)}$$

Extracting root, $x=\sqrt[2n]{\dfrac{s-\frac{1}{c^n}}{\left(s-\frac{1}{a^n}\right)\left(s-\frac{1}{b^n}\right)}}$; and similar results for y and z.

20. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

At what price must the government sell 5% \$100 bonds to run 10 years, interest payable annually (quarterly), to make them equivalent to 3% bonds at par to run 10 years, interest payable annually (quarterly)? At what price if interest be paid semi-annually?

Solution by THEODORE L. DeLAND, Examiner, United States Civil Service Commission, Washington, D. C.

Let the symbols be as follows: $x=$ price paid for \$1 of the 5% bond; $r=3\%$, the rate the investor realizes; $t=$ the time the bond runs in years; $n=$ the number of times the interest is paid in 1 year; $r_1=$ the rate the bond draws per annum.

Following the United States Rule for Partial Payments and we have, *after each interest payment*, *amounts due* as follows:

$$\text{First:—}\quad x\left(1+\frac{r}{n}\right)-\frac{r_1}{n}.$$

$$\text{Second:—}\quad x\left(1+\frac{r}{n}\right)^2-\frac{r_1}{n}\left(1+\frac{r}{n}\right)-\frac{r_1}{n}.$$

$$\text{Third:—}\quad x\left(1+\frac{r}{n}\right)^3-\frac{r_1}{n}\left(1+\frac{r}{n}\right)^2-\frac{r_1}{n}\left(1+\frac{r}{n}\right)-\frac{r_1}{n}.$$

We can now determine the law of the series by inspection; and we also know that after the last interest payment is made there is due and payable the principal of the bond or \$1; and we also know that after the interest and principal are paid there is nothing due. We form the general equation as follows:

$$x\left(1+\frac{r}{n}\right)^{nt}-\frac{r_1}{n}\left(1+\frac{r}{n}\right)^{nt-1}-\frac{r_1}{n}\left(1+\frac{r}{n}\right)^{nt-2}-\ldots-\frac{r_1}{n}\left(1+\frac{r}{n}\right)-\frac{r_1}{n}-1=0.$$

$$\therefore x\left(1+\frac{r}{n}\right)^{nt}=\frac{r_1}{n}\left[\left(1+\frac{r}{n}\right)^{nt-1}+\left(1+\frac{r}{n}\right)^{nt-2}+\ldots+\left(1+\frac{r}{n}\right)+1\right]+1.$$

The function within the brackets is a finite, determinable, equirational descending progression. Reverse the order of the terms of the function to make the series an ascending series, and to escape negative exponents in the result, and we have:

$$x\left(1+\frac{r}{n}\right)^{nt}=\frac{r_1}{n}\left[1+\left(1+\frac{r}{n}\right)+\left(1+\frac{r}{n}\right)^2+\ldots\right.$$

$$\left.+\left(1+\frac{r}{n}\right)^{nt-2}+\left(1+\frac{r}{n}\right)^{nt-1}\right]+1.$$

We observe in the series that the ratio is $\left(1+\frac{r}{n}\right)$, and that the last term is, $\left(1+\frac{r}{n}\right)^{nt-1}$ Sum it and we have:

$$x\left(1+\frac{r}{n}\right)^{nt}=\frac{r_1}{n}\left[\frac{\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)^{nt-1}-1}{\left(1+\frac{r}{n}\right)-1}\right]+1.$$

$$\therefore x\left(1+\frac{r}{n}\right)^{nt}=\frac{r_1}{r}\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]+1.$$

$$\therefore x\left(1+\frac{r}{n}\right)^{nt}=\frac{r_1\left(1+\frac{r}{n}\right)^{nt}}{r}-\frac{r_1}{r}+1.$$

$$\therefore x\left(1+\frac{r}{n}\right)^{nt}=\frac{1}{r}\left[r_1\left(1+\frac{r}{n}\right)^{nt}-(r_1-r)\right].$$

$$\therefore x=\frac{1}{r}\left[r_1-\frac{r_1-r}{\left(1+\frac{r}{n}\right)^{nt}}\right]\dots\dots\dots\dots\dots(\text{A}).$$

The above is the general equation in its simplest form. Substitute values from the problem, observing that $n=1$, and we have:

$$x=\frac{1}{.03}\left[.05-\frac{.05-.03}{(1.03)^{10}}\right]\dots\dots(\text{B}).$$

Releive the equation of decimals and we have:

$$x=\frac{1}{3}\left[5-\frac{2}{(1.03)^{10}}\right]\dots\dots\dots\dots\dots(\text{C}).$$

The numerical computations are as follows:

Log 2=...................................0.3010300.
10 log 1.03=0.01283722×10................0.1283722.
Log^{-1} 1.488188=..........................0.1726578.

5.000000−1.488188=3.511812. From (C), $x=\frac{1}{3}(5-1.488188)=\frac{1}{3}$ of 3.511812 =1.170604.

$\therefore x=\$1.170604$; or $100x=\$117.06$, the price of a \$100-bond.

The general equation (A) can also be applied to the solution of the semi-annual bond. If we apply it to a quarterly bond we solve the government problem which confronted the Secretary of the Treasury when he placed the late \$50,000,000-loan on the market; and which has given rise to so much hostile criticism from newspapers and actuaries. The figures given by Secretary Carlisle that the price, \$117.22$\frac{3}{10}$, for a \$100-bond at 5%, to realize 3% to the investor, is a just equivalent, may be shown as follows:

Substitute data for a quarterly bond from the problem, observing that $n=4$, and we have: $x=\frac{1}{3}\left[5-\frac{2}{(1.0075)^{40}}\right]$

Log 2....................................0.3010300
40 log 1.0075=0.0032451×40= 0.1298040
Log^{-1} 1.48329=........................0.1712260.

$\therefore\ x=\frac{1}{3}(5-1.48329)=\frac{1}{3}$ of $3.51671=1.17223$. $\therefore\ 100x=\$117.22\frac{3}{10}$, the price to be paid for a \$100-bond at 5%, interest payable quarterly for 10 years, to realize 3% per annum payable quarterly.

Also solved by *J. H. DRUMMOND*, *H. C. WHITAKER*, and *G. B. M. ZERR*.

NOTE:—Problem 19 was also solved by *J. H. DRUMMOND*, *M. A. GRUBER*, *T. L. DeLAND*, *J. F. W. SHEFFER*, *G. B. M. ZERR*, and *F. P. MATZ*.

PROBLEMS.

26. **Proposed by ALVIN E. SCHMIDT, Winesberg, Ohio.**

Show that $abc>(a+b-c)(a+c-b)(b+c-a)$ unless $a=b=c$.

27. **Proposed by A. H. BELL, Hillsboro, Illinois. (The problem from H. C. WILKS, Skull Run, Virginia.**

An oarsman in rowing a boat down stream 7 miles from A to B and then back requires 12 minutes longer time, than commencing from B, and rowing up and back; the rate of speed for the 1st half of the time is 5 miles per hour, and for the 2nd half of the time is $4\frac{1}{2}$ miles per hour. Required the current.

28. **Proposed by H. W. DRAUGHON, Clinton, Louisiana.**

The working capacity of a horse is constant between the ages of a and b years, and decreases at a constantly accelerated rate from the age of b years to that of c years, becoming 0 at the latter age. If the value of the horse at the age of a years is d, give a formula for finding his value at any subsequent time.

Solutions to these problems should be received on or before September 1st.

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

14. **Proposed by HENRY HEATON, M. S., Atlantic, Iowa.**

Through two given points to pass four circles tangent to two given circles.

Solution by the PROPOSER.

In the figure, C and C' the intersection of the common tangents to the two circles are known as the external and internal centers of similitude.

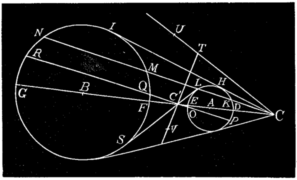
It is not necessary to demonstrate here the following well known properties:

I. If $CKLMN$ be any secant line from C, then $CL\times CM=CK\times CN=CH\times CI$.

II. If $POC'QR$ be any line through C', secant to both circles, $C'O\times C'R=C'Q\times C'P=C'L\times C'S$.

III. If any circle be drawn to which both circles are either internally or externally tangent, the line through the points of tangency will pass through the external center of similitude, C.

IV. If any circle be drawn to which one of the given circles is internally tangent, and the other externally tangent, the line joining the points of tangency will pass through the internal center of similitude, C'.



Let T be the given point. On the line through CT take a point U such that $CT\times CU=CH\times CI$. This may be done by passing a circle through I, T, and H. It will cut CT in U.

The circle to which both circles are either internally or externally tangent and which passes through T, will also pass through U.

This follows from I. and III.

If through U and T we pass two circles tangent to either of the given circles, (Prob. 17.) they will be tangent to the other.

On the line through $C'T$ take a point V such that $C'T\times C'V=C'S\times C'L$. This may be done by passing a circle through T, L, and S. It will cut $C'T$ in V.

The circle to which one of the given circles is tangent internally and the other externally and which passes through T will also pass through V. This follows from II. and IV.

If through V and T we pass a circle to which one of the given circles is internally tangent, the other will touch it externally and *vice versa.*

Excellent solutions received from *Professors G. B. M. ZERR, P. H. PHILBRICK*, and *H. C. WHITAKER.*

15. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A man starts from the center of a circular 10 acre field and walks due north a certain distance, then turns and walks south-west till he comes to the circumference, walking altogether 40 rods. How far did he walk before making the turn?

Solution by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let x be the first distance, y the second, and R the radius, then $x+y=40$....(1) and $x^2+y^2-2xy\cos 45^\circ=R^2$, or $x^2-xy\sqrt{2}+y^2=R^2=\frac{1600}{\pi}$....(2).

Squaring (1) and subtracting (2) from it, we have $xy=800\left(1-\frac{1}{\pi}\right)$ $(2-\sqrt{2})\ldots.(3)$. Now we at once get $y-x=40\sqrt{1-2\left(1-\frac{1}{\pi}\right)(2-\sqrt{2})}\ldots.(4)$.

Combining (1) and (2), we finally obtain

$$x=40\left[1-\sqrt{1-2\left(1-\frac{1}{\pi}\right)(2-\sqrt{2})}\right],\ y=40\left[1+\sqrt{1-2\left(1-\frac{1}{\pi}\right)(2-\sqrt{2})}\right].$$

Also solved by *P. H. PHILBRICK, G. B. M. ZERR, H. M. CASH, P. S. BERG, CHARLES E. MYERS, J. A. CALDERHEAD, SETH PRATT*, and *H. C. WHITAKER*.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

16. Proposed by F. P. MATZ, M. Sc., Ph.D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with regard to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland. and CHARLES E. MYERS, Canton, Ohio.

Let $\frac{2x}{1-x^2}=z$, then $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}z$; but $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ by Trigonometry equals $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}z$. Hence $\frac{dz}{dz}=1$, that is, since both expressions are identical, the first differential coefficent is$=1$.

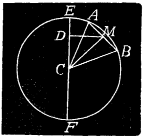
Also solved by Professor *MATZ, SCHMITT*, and *ZERR*.

17. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

To find the volume generated by revolving a circular segment whose base is a given chord, about any diameter as an axis.

Solution by the PROPOSER.

In the circle, center C, draw any diameter ECF, and also any chord



AB. Bisect AB in M, and draw CM, and also MD perpendicular to EF.

Let R be the radius of the circle, and $2a$ the length of the given chord. Put $\angle DCM=A$, and $CM=x$; then $DM=x \sin A$, and $AB=2(R^2-x^2)^{\frac{1}{2}}$.

If AB be revolved about EF, as an axis, the surface generated will be, $AB\times 2\pi DM=4\pi \sin A(R^2-x^2)^{\frac{1}{2}}x$; $\therefore$ the required volume is,

$$V=4\pi \sin A \int_{(R^2-x^2)^{\frac{1}{2}}}^{R}(R^2-x^2)^{\frac{1}{2}}x\, dx=\frac{4}{3}\pi \sin A.a^3.$$

This expression is independent of the radius of the circle. $\therefore$ The same volume will be generated by the segment of any circle, chord $2a$ and angle A remaining constant. When $\angle A=90^\circ$, $V=\frac{4}{3}\pi a^3$.

Professor Arthur E. Haynes gives a solution of this particular case of the problem, in *The Mathematical Magazine*, Vol. II. No. 6, page 83. If the segment is cut by the axis, the formula will, of course, give the difference between the volumes generated by the two divisions of the segment. $\therefore$ When $a=R$, we have, $V=\frac{4}{3}\pi \sin AR^3$, the volume generated by any sector, angle at the center being $2A$. When $A=30^\circ$, $V=\frac{2}{3}\pi a^3$. In this case the volume is just $\frac{1}{2}$ as great as when $\angle A=90^\circ$.

Many other interesting cases will suggest themselves; those that I have considered will give some idea of the comprehensiveness of the formula.

Also solved by *F. P. MATZ*, *C. E. MYERS*, *J. F. W. SCHEFFER*, and *G. B. M. ZERR*.

Professor *J. C. NAGLE*, B. Sc., M. A., Department of Civil Engineering and Physics, Agricultural and Mechanical College of Texas, College Station, Texas, sent an excellent solution to prob. 8, after the April No. had gone to press.

PROBLEMS.

22. **Proposed by MOSES C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Indiana.**

Solve the following Differential Equation:—

$$(6x^3+20x^2-2x)\frac{d^2y}{dx^2}-(9x^2+10x+1)\frac{dy}{dx}+(1+9x)y=0.$$

23. **Proposed by W. I. TAYLOR, Baldwin University, Berea, Ohio.**

From a point O situated in the plane of a plane curve, radii vectores are drawn to different points of the curve, and on each one a distance is laid off from O inversely proportional to the length of the radius vector; to determine the tangent at any point of the locus of the points thus obtained. [*Byerly's Diff. Calculus, p. 177, ex. 1.*]

24. Proposed by C. W. M. BLACK, A. M., Department of Mathematics, Wilmington Conference Academy, Dover, Delaware.

At the President's reception, the citizens are admitted at 12:30, but the line begins to form at 11 in front of the gate. By the time the doors are opened, there are in line 5400 people, who have gathered at a rate per second proportional to the time after 11. The President is able to receive them at the rate of 45 per second. At what time should a person have joined the line to get through with the least delay?

Solutions to these problems should be received on or before September 1st.

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

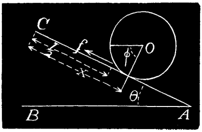
SOLUTIONS TO PROBLEMS.

7. Proposed by DeVOLSON WOOD, M. A., M. Sc., C. E., Professor of Engineering, Stevens Institute of Technology, Hoboken, New Jersey.

A hollow sphere filled with frictionless water rolls down a rough plane whose length is l and inclination θ; when half way down the water suddenly freezes and adheres to the sphere. Required the time of the decent.

Solution by J. C. NAGLE, M. A., C. E., M. C. E., Professor of Civil Engineering and Physics, Agricultural and Mechanical College, College Station, Texas.

Let the mass of water be m and the sphere a shell without weight. Then before freezing of the water the motion will be equivalent to simple sliding on a rough inclined plane.



Take origin at C, the point at which motion begins and let x be measured along the plane. Let f be the friction acting up the plane. The equation of motion for x less than $\frac{l}{2}$ is: $m\frac{d^2x}{dt^2}=mg\sin\theta-f$....(1),

from which, by integration, $V=\frac{dx}{dt}=g\sin\theta.t-\frac{f}{m}t$....(2).

and $x=g\sin\theta t^2-\frac{f}{m}t^2$....(3).

When $x=\frac{l}{2}$, $t=T\sqrt{\frac{ml}{2(mg\sin\theta-f)}}$.

This value of t in (2) determines the value of $v=V$ at the instant the water freezes.

Now let v' be velocity of center immediately after freezing and ω the angular velocity about point of contact, which is the same as if center be regarded as fixed and the sphere revolving about that point.

Call the radius of gyration of the sphere K and we have for the effect of the impulse, since $K^2=\frac{2}{5}r^2$ for a homogensus sphere:

$$m(V-v')r=m.\frac{2}{5}r^2.\omega\ldots.(4)$$

and as the point of contact between sphere and plane has no velocity,

$$v'=r\omega\ldots.(5).$$

From (4) and (5) $\omega=\frac{5V}{7r}$, $v'=\frac{5}{7}V.$

If ϕ be the angle through which the sphere has turned after freezing of the water, at time t. Then $x=r\phi+\frac{l}{2}\ldots.(6)$.

The acceleration along plane is given by (1) and for the motion of rotation we have $mK^2\frac{d^2\phi}{dt^2}=fr\ldots.(7)$.

From (1) and (7) $mr\frac{d^2x}{dt^2}+mK^2\frac{d^2\phi}{dt^2}=mrg\sin\theta\ldots.(8)$

and from (6) $\frac{d^2x}{dt^2}=r\frac{d^2\phi}{dt^2}$, which in (8) gives

$$\frac{d^2x}{dt^2}=\frac{r^2}{r^2+K^2}g\sin\theta=\frac{5}{7}g\sin\theta\ldots.(9).$$

Then will $v=\frac{dx}{dt}=\frac{5}{7}g\sin\theta\ldots.(10)$

and $x=\frac{5}{14}g\sin\theta t^2+v't+C\ldots.(11)$.

When $x=\frac{l}{2}$, $t=T$ and therefore $C=\frac{l}{2}-\frac{5}{14}g\sin\theta T^2-v'T.$

(11) now becomes $x=\frac{5}{14}g\sin\theta t^2+v't+\frac{l}{2}-\frac{5}{14}g\sin\theta T^2-v'T.$

Putting $x=l$ and solving for $t=t'$, the time of decent, gives:

$$t'=\frac{7v'}{5g\sin\theta}+\sqrt{\frac{7l}{5g\sin\theta}+\frac{49v'^2}{25g^2\sin^2\theta}+\frac{14vT}{5g\sin\theta}+T^2}.$$

8. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

A heavy bar AB of length a falls about its lower end B from a vertical to a horizontal position, when the end A is suddenly fixed and B is set free, so that the bar falls into a vertical position AB as at first; then A is set free, and B is fixed, so that the bar again falls about B into a horizontal position, when the end A is suddenly fixed, and B is set free and so on; find the angular velocity w of the bar about the upper end, when it takes a vertical position for the nth time.

[Selected from *Price's Infinitesimal Calculus.*]

Solution by the PROPOSER.

Let m be the mass of the rod and I its moment of inertia about one end. Let w_1, w_2, w_3, etc. be the angular velocities immediately before the successive impacts; w'_1, w'_2, w'_3, etc. the angular velocities immediately after.

When the rod reaches a horizontal position for the first time its *vis viva* is $Iw^2{}_1$; the work done by gravity, $\frac{1}{2}a.mg$.

$$\therefore\ Iw^2_1=mag,\ \tfrac{1}{3}ma^2w^2_1=mag,\ w^2_1=\frac{3g}{a}\,.$$

If r is the distance from the end of the rod now set free to any element of length dr,

$$w'_1=\frac{\text{moment of impulses}}{\text{mement of inertia}}=\frac{\int_0^a(a-r)r.w_1dr}{I}=\frac{w_1}{2},$$ the density and cross-section of the rod being taken as unity.

The principle of *vis viva* gives $I(w^2_2-w'^2_1)=mag=Iw^2_1$.

$\therefore\ w^2_2=\dfrac{5}{4}w^2_1=\dfrac{15g}{4a}$. w^2 is the angular velocity with which the rod reaches a vertical position for the first time. The impact which now occurs causes a diminution of the angular velocity of one half, so that $w'_2=\dfrac{w_2}{2}$.

As before, $I(w^2_3-w'^2_2)=mag=Iw^2_1$.

$\therefore\ w^2_2-w'^2_2=w^2_1$ which, by substitution, gives $w^2_3=\dfrac{63g}{16a}$.

Similarly, just before the next impact, when the rod becomes vertical the second time, $w^2_4=\dfrac{255g}{64a}$.

Again, $w^2_6=\dfrac{4095g}{1024a}$, and so on.

The law of increase can be seen from the following:—

(Angular velocity)^2 when rod reaches vertical 1st time

$$=\omega^2{}_2=\frac{15g}{4a}=\frac{4g}{a}\cdot\frac{15}{16}.$$

(Angular velocity)^2 when rod reaches vertical 2nd time

$$=\omega^2{}_4=\frac{4g}{a}\cdot\frac{255}{256}.$$

(Angular velocity)^2 when rod reaches vertical 3rd time

$$=\omega^2{}_6=\frac{4g}{a}\cdot\frac{4095}{4096}.$$

....

$\therefore$ (Angular velocity)^2 when rod reaches vertical nth time

$$=\frac{4g}{a}\cdot\frac{4^{2n}-1}{4^{2n}}=\frac{4g}{a}\left[1-\left(\frac{1}{4}\right)^{2n}\right].$$

The result given by Price is $\frac{4g}{a}\left[1+\left(\frac{1}{4}\right)^{4n-2}\right]$.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTION TO THE CELEBRATED INDETERMINATE EQUATION.

$$x^2-Ny^2=\pm1.$$

By A. H. BELL, Hillsboro, Illinois.

[Continued from the May Number.]

Well known Formulas employed to shorten the computations.

The No. of Terms of the series is nearly equal to $(D_1\div D_2+1)2=(J)$.

The convergent Series for the nth Term is, $S_n=S_1+(M-1)(n-2)\frac{D_2}{2}+(n-1)D_1=(K)$.

The Series of the m^s and n^s for the nth Term is; $M_n=M_1+(n-1)$. Com. Diff. $=(L)$.

Process: Having the number of Terms by (J); obtain the conv. Diff. for $(n-1)^{th}$ Term by (K): then add the D_1 or the 1st order of Difference for this Term will give the nth term; if the two convergent differences are of

contrary signs. The necessary limit or end of the Series for our purposes is complete; the D_1 to be added is the same as $(L)=D_n=D_1+(n-1)D^2=(M)$.

Example $x^2-61y^2=\pm1$. To find x and y. Some new features will be introduced in this example, and only the actual work is shown.

Series A.				
No. of Terms	1st	2nd......4th		5th.
Conv. Diff.	+15	−3......−9		+5.
$D_1 = -18$				
$D_2 = +8$				

Series B. No.	1	2	3	4
			$+A.5$	
Name of term		C.Diff. $+A.4$		
Conv. Diff.	+4	−9	−12	−5
D_1	−13			
D_2	+10			

Series C. No.	1	2.........4	5
		$+B.4$	
Name	$B.3+A.5$		
Conv. Diff.	−12	+5.........+9	−4
D_1	+17		
D_2	−10		

Series D. No.	1	2.........4	5
		$+C.5$	
	$B.4+C.4$		
Diff.	−5	+9 +13	+3
	+14		
	−8		

Series E. No.	1	2..............6
		$+D.5$
	$D.4+C.5$	
Conv. Diff.	+13 −4	+1
	−17	
	+6	

m and n, in column.

Operation.	m^s	n^s
Com. Diff. =	5......	39.
	×3	×3
	15	117
Add...........	1........	8
Series A. 4, =	16.....	125
Series A. 5, =	21.....	164
	×1	
Add A,4 = B.3 =	37.....	289
B.4 =	58....	453
	×2	
	116	906
Add	21	164
C. 4 =	137....	1070
C. 5 =	195....	1523
	×2	
	390	3046
Add	137	1070
D. 4 =	527....	4116
D. 5 =	722....	5639
	×5	
	3610	28195
Add C, 5 =	195	1523
E. 6. =	3805.m^s.	29718.n^s

∴ we have the −1. condition. The +1. condition is by formula (A). These are the 1st. and least values that can be given.

The amount of work required in this class of numbers N, is equivalent to summing two cycles of continued fraction.

From Legendre's Table, date 1798 $x^2-991y^2=\pm1$

x=397516,400906,811930,638014,896080.

y=12055,735790,331359,447442,538767.

In the celebrated "Cattle Problem" of Archimedes,

$x^2-410,286423,278424$, $y^2=+1$ x not being required. The Hillsboro, Ill. Mathematical Club, make y=185892,190138,691330,945825,420863.... 103224 digits....710208,663490. Making a number $\frac{1}{4}$ mile long.

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

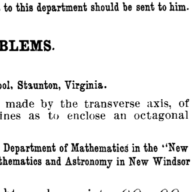
SOLUTIONS TO PROBLEMS.

4. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Four points taken at random in each half, made by the transverse axis, of an ellipse, are joined in such a way by straight lines as to enclose an octagonal surface; find the mean area of this surface.

II. Solution by F. P. MATZ, M. Sc., Ph. D., Editor of the Department of Mathematics in the "New England Journal of Education", and Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let P_1, $P_2,\ldots,P_8$ represent the eight random points; OQ_1, OQ_2, $\ldots,OQ_8$ the radius vectors drawn through these points; $x_1=OP_1$, $x_2=OP_2\ldots,x_8=OP_8$ the distances of the random points from the center of the ellipse. The random points may range over the radius vectors on which they lie. The point Q_1 may range over the elliptic arc AQ_2; that is, the number of radius vectors on which P_1 may lie is proportional to the length of the elliptic arc AQ_2. The points Q_2, $Q_3,\ldots,Q_8$ may range respectively over the elliptic arcs AQ_3, AQ_4, AA', $A'Q_6$, $A'Q_7$, $A'Q_8$, $A'A$. Represent the polar co-ordinates of the point Q_1, $Q_2,\ldots,Q_8$ by $(r_1,\ \theta_1)$, $(r_2,\ \theta_2)\ldots,(r_8,\ \theta_8)$; then area of the octagonal surface $P_1P_2\ldots P_8P_1=A=$ the sum of the area of the eight triangles P_1OP_2, $P_2OP_3,\ldots,P_8OP_1$; that is, $A=\frac{1}{2}[x_1x_2\sin(\theta_2-\theta_1)+x_2x_3\sin(\theta_3-\theta_2)+\ldots+x_8x_1\sin(\theta_1-\theta_8)]$.



Representing the specified *elliptic* arcs by l_1, $l_2,\ldots,l_8$, the required mean area becomes

$$A=\frac{1}{\triangle}\int_0^{l_8}\int_0^{r_8}\int_0^{l_7}\int_0^{r_7}\int_0^{l_6}\int_0^{r_6}\int_0^{l_5}\int_0^{r_5}\int_0^{l_4}\int_0^{r_4}\int_0^{l_3}\int_0^{r_3}\int_0^{l_2}\int_0^{r_2}\int_0^{l_1}\int_0^{r_1} A\,ds_8dx_8\times$$

$$ds_7dx_7ds_6dx_6ds_5dx_5ds_4dx_4ds_3dx_3ds_2dx_2ds_1dx_1\ldots(1), \text{ in which}$$

$$\triangle=\int_0^{l_8}\int_0^{r_8}\int_0^{l_7}\int_0^{r_7}\int_0^{l_6}\int_0^{r_6}\int_0^{l_5}\int_0^{r_5}\int_0^{l_4}\int_0^{r_4}\int_0^{l_3}\int_0^{r_3}\int_0^{l_2}\int_0^{r_2}\int_0^{l_1}\int_0^{r_1} ds_8dx_8\times$$

$$ds_7dx_7ds_6dx_6ds_5dx_5ds_4dx_4ds_3dx_3ds_2dx_2ds_1dx_1.$$

From the ellipse, as per *Conic Sections*, r^2_1, $r^2_2,\ldots,r^2_8=$

$\dfrac{b^2}{1-e^2\cos^2\theta_1}$, $\dfrac{b^2}{1-e^2\cos^2\theta_2},\ldots,\dfrac{b^2}{1-e^2\cos^2\theta_8}$; and the superior integral limits of

$x_1, x_2, \dots, x_8$, as obtained from this system of equations.

Differentiating these equations, we have respectively,

$$\left(\frac{dr_1}{d\theta_1}\right)^2, \left(\frac{dr_2}{d\theta_2}\right)^2, \dots, \left(\frac{dr_8}{d\theta_8}\right)^2 = \frac{b^2e^4\sin^2\theta_1\cos^2\theta_1}{(1-e^2\cos^2\theta_1)^3}, \frac{b^2e^4\sin^2\theta_2\cos^2\theta_2}{(1-e^2\cos^2\theta_2)^3},$$

$$\dots, \frac{b^2e^4\sin^2\theta_8\cos^2\theta_8}{(1-e^2\cos^2\theta_8)^3} \dots (2).$$

By means of the formula for the rectification of plane curves represented by polar co-ordinates, we have from (2)

$$\int_0^{l_1} ds_1 = b\int_0^{\theta_2} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_1]}}{(1-e^2\cos^2\theta_1)^{\frac{3}{2}}} d\theta_1;$$

$$\int_0^{l_2} ds_2 = b\int_0^{\theta_3} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_2]}}{(1-e^2\cos^2\theta_2)^{\frac{3}{2}}} d\theta_2;$$

$$\int_0^{l_3} ds_3 = b\int_0^{\theta_4} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_3]}}{(1-e^2\cos^2\theta_3)^{\frac{3}{2}}} d\theta_3;$$

$$\int_0^{l_4} ds_4 = b\int_0^{\pi} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_4]}}{(1-e^2\cos^2\theta_4)^{\frac{3}{2}}} d\theta_4;$$

$$\int_0^{l_5} ds_5 = b\int_0^{\theta_6} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_5]}}{(1-e^2\cos^2\theta_5)^{\frac{3}{2}}} d\theta_5;$$

$$\int_0^{l_6} ds_6 = b\int_{\pi}^{\theta_7} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_6]}}{(1-e^2\cos^2\theta_6)^{\frac{3}{2}}} d\theta_6;$$

$$\int_0^{l_7} ds_7 = b\int_{\pi}^{\theta_8} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_7]}}{(1-e^2\cos^2\theta_7)^{\frac{3}{2}}} d\theta_7;$$

$$\int_0^{l_8} ds_8 = b\int_{\pi}^{2\pi} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_8]}}{(1-e^2\cos^2\theta_8)^{\frac{3}{2}}} d\theta_8.$$

The evaluation of the thirty-two integrals indicated in (1) is a labor sufficient to discourage even a mathematical Hercules.

6. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Find the average length of all the diameters that can be drawn in a given ellipse.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let $2r$ represent any diameter; then from the *central-polar* equation of the ellipse, $r^2 = \dfrac{b^2}{1-e^2\cos^2\theta}$, we have $2r = \dfrac{2b}{\sqrt{(1-e^2\cos^2\theta)}}$,

$$\frac{dr}{d\theta} = \frac{be^2\sin\theta\cos\theta}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}; \text{ and } \frac{ds}{d\theta} = b\sqrt{\left(\frac{1-e^2(2-e^2)\cos^2\theta}{(1-e^2\cos^2\theta)^3}\right)}.$$

Since the number of diameters that can be drawn in an elliptic quadrant is proportional to the length of the elliptic arc bounding that quadrant, the required average length becomes

$$D=2b\int_0^{\frac{1}{2}\pi}\frac{\surd[1-e^2(2-e^2)\cos^2\theta]}{(1-e^2\cos^2\theta)^2}\,d\theta\div\int_0^{\frac{1}{2}\pi}\frac{\surd[1-e^2(2-e^2)\cos^2\theta]}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}\,d\theta.$$

Representing $e^2(2-e^2)$ by p and expanding,

$$D=2b\int_0^{\frac{1}{2}\pi}\frac{(1-\frac{1}{2}p\cos^2\theta-\frac{1}{8}p^2\cos^4\theta-\frac{1}{16}p^3\cos^6\theta-\frac{5}{128}p^4\cos^8\theta-\text{etc.})}{(1-e^2\cos^2\theta)^2}d\theta$$

$$\div\int_0^{\frac{1}{2}\pi}\frac{(1-\frac{1}{2}p\cos^2\theta-\frac{1}{8}p^2\cos^4\theta-\frac{1}{16}p^3\cos^6\theta-\frac{5}{128}p^4\cos^8\theta-\text{etc.})}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$=2b\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}e^2(2+e^2)\cos^2\theta+\tfrac{1}{8}e^4(4+12e^2-e^4)\cos^4\theta+\tfrac{1}{16}e^6(-8+52e^2$$

$$-10e^4+e^6)\cos^6\theta+\tfrac{1}{128}e^8(-272+800e^2-264e^4+56e^6-5e^8)\cos^8\theta+\text{etc.}]d\theta$$

$$\div\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}e^2(1+e^2)\cos^2\theta+\tfrac{1}{8}e^4(-1+10e^2-e^4)\cos^4\theta+\tfrac{1}{16}e^6(-15+39e^2$$

$$-9e^4+e^6)\cos^6\theta+\tfrac{1}{128}e^8(-261+564e^2-222e^4+28e^6-5e^8)\cos^8\theta+\text{etc.}]d\theta$$

$$=2b\left(\frac{1+\frac{1}{4}e^2(2+e^2)+\frac{3}{64}e^4(4+12e^2-e^4)+\frac{5}{256}e^6(-8+52e^2-10e^4+e^6)}{1+\frac{1}{4}e^2(1+e^2)+\frac{3}{64}e^4(-1+10e^2-e^4)+\frac{5}{256}e^6(-15+39e^2-9e^4+e^6)}\right),$$

which is the required average length.

Cor.—Put $e=\frac{1}{2}$; then substitute and reduce, we obtain

$D=\dfrac{1218749}{1133057}$ of $2b=1.07563$ times $2b=1.076\times2b=\dfrac{269}{250}$(of the minor axis of the given ellipse).

This problem was also solved by Professors *SCHEFFER* and *ZERR*. Professor *MATZ* sent in three different solutions.

PROBLEMS.

13. Proposed by I. L. BEVERAGE, Monterey, Virginia.

Find the mean values of the roots of the quadratic $x^2-ax+b=0$, the roots being known to be real, but b being unknown and positive.

14. Proposed by CHARLES E. MYERS, Canton, Ohio.

$\frac{1}{3}$ of all the mellons in a patch are not ripe, and $\frac{1}{4}$ of all the mellons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a mellon at random, what is the probability that he will get a good one?

15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\surd2-1),=.236+$." Is this result perfectly correct as to fact?

16. Proposed by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

What is the average volume common to a cube and a rectangular solid one inch square, the axis of rectangular solid being equal to and coinciding with the diagonal of the cube?

Solutions to these problems should be received on or before September 1st.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

8. Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Find a general expression for the (integral) co-ordinates of a triangle with sides of integral lengths.

Solution by the PROPOSER.

My own method of solving this problem has been to take the three equations $y=\frac{a}{b}x$, $y=\frac{c}{d}x$, and $y=\frac{e}{f}x+g$, and eliminating x and y solve for g; a and b, c and d, e and f being $\pm$ sides of right triangles.

The sides I usually take for triangles have lengths (39, 34, 25), (13, 45, 40), (10, 39, 35). I am in the habit of giving a whole group of problems with the same triangle to be worked out consecutively; *e. g.*, Find, (1), length of each side; (2) equations of each side, (3), length each altitude; (4), sine each angle; (5), area; (6), equation of bisectors of each angle; (7), position of centers of inscribed and escribed circles; (8), their radii, and so on.

Another day I give something like this; A $\triangle$ has its vertices at (5, 10), (6, 4) and (3, 2), find, (1), the equations of the sides; (2), the equations of the altitudes; (3), the point of intersection of the altitudes; (4), the co-ordinates of the middle point of each side; (5), the equations of the medians; (6), the intersection of same; (7), the equations of perpendiculars through middle points of each side; (8), their intersection; (9),the equation of line through (3), (6) and(8).

9. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Wires of five different metals A, B, C, D, E, having resistances a, b, c, d, e, have their ends soldered together at two junctions which are maintained at different constant temperatures. If the strength of current in E, when all five wires are continuous, is S, the strength of current when B, C, D, are cut is S_a, the strenth of current when A, C, D, are cut is S_b, the strength of current when A, B, D, are cut is S_c, find the strength of current S_x, when A, B, C, are cut.

Solution by the PROPOSER.

Let l, m, n, o, p, q be the potentials of the wires and solder at the junction P. l', m', n', o',p',q' the potentials of the wires and solder at the junction Q.

Let v, w, x, y, S be the currents in A, B, C, D, E supposed to be going from P to Q.

The electromotive force in the wire A is $(l'-q')-(l-q)$ and by Ohm's law this is equal to the product of the resistance into the current.

$\therefore$ $(l'-q')-(l-q)=av$ or $l'-l-av=q'-q$.

Similarly $m'-m-bw=q'-q$,

$n'-n-cx=q'-q$,

$o'-o-dy=q'-q$,

$p'-p-eS=q'-q$.

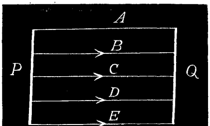
Also $v+w+x+y+S=0$ by symmetry.

Similarly $l'-l+aS_a=p'-p-eS_a$,

$m'-m+bS_b=p'-p-eS_b$,

$n'-n+cS_c=p'-p-eS_c$,

$o'-o+dS_x=p'-p-eS_x$.



Therefore

$eS-av=(a+e)S_a$ or $bcdeS-abcdv$

$=bcd(a+e)S_a$,

$eS-bw=(b+e)S_b$ or $acdeS-abcdw$

$=acd(b+e)S_b$,

$eS-cx=(c+e)S_c$ or $abdeS-abcdx=abd(c+e)S_c$,

$eS-dy=(d+e)S_x$ or $abceS-abcdy=abc(d+e)S_x$.

Therefore $(bcde+acde+abde+abce)S-abcd(v+w+x+y)=bcd(a+e)S_a$
$+acd(b+e)S_b+abd(c+e)S_c+abc(d+e)S_x$ but $v+w+x+y=-S$.

$\therefore$ $S_x=[(abcd+bcde+acde+abde+abce)S-abcd(S_a+S_b+S_c)-bcdeS_a$
$-acdeS_b-abdeS_c]\div abc(d+e)$.

PROBLEMS.

13. Proposed by CHARLES E. MYERS, Canton, Ohio.

A soap bubble 2 inches in diameter, is filled with one part of hydrogen gas and 15 parts of air. If the bubble just floats in the air, find the thickness of the film.

14. Proposed by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

I have a glass paper-weight in the form of a regular icosahedron. I let the sun's rays fall upon it, at various angles, also upon one of the vertices. How many complete spectra will be formed? How many will be of white light? What position will give maximum number of spectra?

15. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates Co., New York.

Required the illuminated area of the Moon's disc when $\frac{2}{3}$ through its first quarter, or 60° of longitude east of the Sun, the Earth and Moon being at their mean distances.

Solutions to these problems should be received on or before September 1st.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

QUADRATURE OF THE CIRCLE.

By EDWARD J. GOODWIN, Solitude, Indiana.

Published by the request of the author.

A circular area is equal to the square on a line equal to the quadrant of the circumference; and the area of a square is equal to the area of the circle whose circumference is equal to the perimeter of the square.

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To quadrate the circle is to find the side of a square whose perimeter equals that of the given circle; rectification of the circle requires to find a right line equal to the circumference of the given circle. The square on a line equal to the arc of 90° fulfills both of the said requirements.

It is impossible to quadrate the circle by taking the diameter as the linear unit, because the square root of the product of the diameter by the quadrant of the circumference produces the side of a square which equals 9 when the quadrant equals 8.

It is not mathematically consistent that it should take the side of a square whose perimeter equals that of a greater circle to measure the space contained within the limits of a less circle.

Were this true, it would require a piece of tire iron 18 feet to bind a wagon wheel 16 feet in circumference.

This new measure of the circle has happily brought to light the ratio of the chord and arc of 90°, which is as 7:8; and also the ratio of the diagonal and one side of a square, which is as 10:7. These two ratios show the numerical relation of diameter to circumference to be as $\frac{5}{4}:4$.

Authorities will please note that while the finite ratio ($\frac{5}{4}:4$) represents the area of the circle to be more than the orthodox ratio, yet the ratio (3.1416) represents the area of a circle whose circumference equals 4 two % greater than the finite ratio ($\frac{5}{4}:4$), as will be seen by comparing the terms of their respective proportions, stated as follows: $1:3.20::1.25:4$, $1:3.1416::1.2732:4$.

It will be observed that the product of the extremes is equal to the product of the means in the first statement, while they fail to agree in the second proportion. Furthermore, the square on a line equal to the arc of 90° shows very clearly that the ratio of the circle is the same in principle as that of the square. For example, if we multiply the perimeter of a square (the sum of its sides) by $\frac{1}{4}$ of one side the product equals the sum of two sides by $\frac{1}{2}$ of one side, which equals the square on one side.

Again, the number required to express the units of length in $\frac{1}{4}$ of a right line, is the square root of the number representing the squares of the linear unit bounded by it in the form of a square whose ratio is as $1:4$.

These properties of the ratio of the square apply to the circle without an exception, as is further sustained by the following formula to express the numerical measure of both *circle* and *square*.

Let C represent the circumference of a circle whose quadrant is *unity*, Q ½ the quadrant, and CQ^2 will apply to the numerical measure of a circle and a square.

We are now able to get the true and finite dimensions of a circle by the exact ratio $\frac{5}{4}:4$, and have simply to divide the circumference by 4 and square the quotient to compute the area.

"An Unreasonable Rule." By SETH PRATT, C. E., Tecumseh, Nebraska.

Professor Philbrick is mistaken as the following Table of convergences of two meridians $69\frac{1}{6}$ miles apart at the equator and extending to the pole for parallels 1° of difference will show. We have R: diff. of cosines :: $69\frac{1}{6}$: convergences.

The object of the Rule is to conform to the law, which requires that "the east and west boundaries of the townships shall conform to the true meridians, and that north and south boundaries shall run on paralles of latitude and, further, that the townships shall be 6 miles square as "nearly as may be."

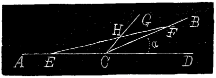
It is readily seen that the distances between any two meridians diminishes as the co-sines of the latitudes, and that corrections should be applied at less distances from any given parallel extending toward the pole, than in a contrary direction. Hence the Rule is "Reasonable."

Lat.	Convergence.	Lat.	Convergence.
0° 1°	0.845ch.	50° 51°	73.786
10° 11°	17.323	60° 61°	84.054
20° 21°	33.819	70° 71°	91.034
30° 31°	49.010	80° 81°	95.249
40° 41°	62.619	89° 90°	96.570

LOBATCHEWSKY'S TRIANGLE. By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Missouri.

"There may be a triangle whose angle sum differs from a straight angle" (two right angles) "by less than any given finite angle however small."

If there may be such a triange the possibility of its construction must be granted. Then let us construct this triangle and call it ECH.



According ho hypothesis this triangle ECH is "a triangle" whose angle sum is equal to two right angles minus the angle α which is "less than any given finite angle however small."

Extend the line EC both ways to

A and D, and construct the angle ECB equal to the angle sum of the triangle ECH, or two right angles$-a$. Then, the supplement of ECB is DCB $=a$. The individual angle DCB is the difference between the finite angle ECF less than two right angles and two right angles and is, therefore, finite.

But the angle DCB by hypothesis is less than any finite angle. Hence, contradictory marks are attributed to the supplementary angle DCB or a, which is absurd. Since the conclusion is absurd, the hypothesis from which it is deduced must be unsound.

"REMARKS ON DIVISION." By WILLIAM F. BRADBURY, Cambridge, Massachusetts.

I take issue with Mr. Ellwood in many of his statements in this article.

1st "If a given product is \$20, and the multiplier 4, we cannot by mere subtraction, find the multiplicand." But we can. Try some number say 4 and subtract thus: $20-4=16$: $16-4=12$: $12-4=8$: $8-4=4$. Now we have subtracted 4 times but have a remainder. Try 5 thus: $20-5=15$: $15-5=10$: $10-5=5$: $5-5=0$. Thus we find a number (5) which subtracted four times (as above) becomes 0. Therefore \$5 is the multiplicand.

Therefore *all* that he says about *abstract* and *concrete* falls to the ground.

2nd. So too $\frac{1}{4}$ of $\frac{8}{9}$ has been called by mathematicians from time immemorial a *compound fraction*. Therefore, this *is the name of it*. And further it is an example in multiplication of fractions. $\frac{1}{4}$ of $\frac{8}{9}=\frac{1}{4}\times\frac{8}{9}$. Now this sign, $\times$, is the sign of multiplication. It is *not* an example in division of fractions. It is an old story that multiplying by a number less than a unit gives a product that is less than the multiplicand. Because multiplying a number by $\frac{1}{4}$ is the same as dividing that number by 4. I do not call multiplying by $\frac{1}{4}$ an example in division. It is a mere quibble in words. What has been so named may as well keep its name.

3rd. So $\dfrac{4}{\frac{2}{3}}$ *is* a complex fraction, because *this is its name*. A fraction may well be defined as an indicated division. Thus $\frac{8}{9}$ may be read eight-ninths, or eight divided by nine. I am perfectly willing to leave $\dfrac{4}{\frac{2}{3}}$ and other complex fractions "as special gifts from high."

EDITORIALS.

Professor Leonard E. Dickson goes to Chicago University as Fellow in Pure Mathematics, having resigned a Shattuck scholarship at Harvard to accept the Fellowship in Chicago University.

All of our subscribers who have not received all of the 7 Nos. of the MONTHLY will please write to the publishers at once, and tell them which Nos. you failed to receive.

Any contributor desiring printed copies of his solution may obtain the same at the rate of $1.00 per hundred pages.

Professor Robert J. Aley, Professor of Mathematics in the Indiana University, is teaching mathematics in a three weeks Institute held at Council Bluffs, Iowa. Professor Aley will teach mathematics in the Leland Sanford Jr. University the coming year, he having exchanged places with Professor John A. Miller.

We take this opportunity to thank our friends for the numerous Invitations, Programs, and Catalogues which we have received. We are glad to receive Catalogues from the various Schools and Colleges as we ourselves are interested in school work.

The July number of *The Monist*, a Chicago quarterly magazine, contains as its leading article, an essay by Dr. George Bruce Halsted which is attracting great attention. It is entitled "The Non-Euclidean Geometry Inevitable," and its new results are of high importance both in the history and philosophy of that facinating subject, with which Dr. Halsted's name has become identified.

BOOKS AND PERIODICALS.

The Principles of Elliptic and Hyperbolic Analysis. By Alexander Macfarlane, M. A., D. Sc., LL. D., Fellow of the Royal Society of Edinburgh, Professor of Physics in the University of Texas, Austin, Texas. Pamphlet form, 47 pp. Price, $0.50. New York: B. Westleman & Co.

In this paper, the author considers the versor part of space more fully and extends the investigation to elliptic and hyperbolic versors. The fundamental theorem of trigonometry is investigated for the sphere, the ellipsoid of revolution and the general ellipsoid; then for the equilateral hyperboloid of two sheets, the equilateral of one sheet and the general hyperboloid. The principles thus deduced are applied to find the complete form of other theorems in spherical trigonometry and to deduce the generalized theorems for the ellipsoid and hyperboloid. At the end, the analogues of rotation theorem are deduced. B. F. F.

Mathematical Teaching and its Modern Methods. By Thurman Henry Saxfford, Ph. D., Field Memorial Professor of Astronomy in Williams College. 8 vo., paper back, 47 pp. Price $0.25. Boston: D. C. Heath & Co.

This valuable little monograph is full of mature thought on the subject of Mathematical Teaching and its Methods. It gives a very careful exposition of the principles that underlie the teaching of arithmetic, algebra, geometry; plane, algebraic, analytical and spherical trigonometry; and the differential and integral calculus, and higher analysis. B. F. F.

The First Steps in Algebra. By G. A. Wentworth, A. M., author of a Series of Text-books in Mathematics. 8 vo., half leather, 184 pp. Price, $0.70. Boston and Chicago: Ginn & Co.

This excellent little work is designed for pupils in the upper grades of gram-

mar schools. The author has kept constantly in view the capacity and taste of the pupils of these grades and has given them an elementary algebra that will please and instruct them.

The introduction of this work in the grammar schools of this country will go far towards dispelling the distaste and fear, which so many pupils have for algebra. The author, in endeavoring to smooth the path of the learner, has not sacrificed pedagogical or psychical principles, the explanation san dmodel solutions being given in a way to preserve the educational value of the study. We heartily recommend it to all teachers of grammar grades as a book well suited to their needs B. F. F.

Elementary Algebra for the use of Preparatory Schools. By Charles Smith, M. A., Author of Treatise on Algebra, etc. Revised and Adapted to American Schools by Irving Stringham, Ph. D., Professor of Mathematics and Dean of the College Faculties, in the University of California. Crown 8 vo., 408 pp. Price, $1.10. New York: Macmillan & Co.

The transition from the traditional algebra of many of our secondary schools to the reconstructed algebra of the best American Colleges is more abrupt than is necessary or creditable. This lack of articulation between the work of the schools and colleges emphasizes the need of a fuller and more thorough course in elementary algebra than is furnished by the text-books now most commonly used. It is with the hope of supplying this new demand that an American edition of Charles Smith's *Elementary Algebra* is published; a work whose excellencies, as represented in former editions, have been recognized by able critics on both sides of the Atlantic. *Preface.*

The author and the reviser of this work need no introduction to the educational public, both having written some valuable mathematical works previously. The book before us is thoroughly Americanized and well adapted to connect, without sudden transition, the course of study in Algebra pursued in the High Schools and Academies with that in our best Colleges. B. F. F.

Elementary Lessons in Physics. By John B. Gifford, Superintendent of Schools, Peabody, Massachusetts. 8 vo. cloth, 169 pages. Price, $0.60 Boston & Chicago: Thompson, Brown & Co.

This little book is intended for the higher classes in Grammar Schools and for High Schools. The experiments are arranged in order of difficulty beginning with the most simple. The book tells the pupil nothing which he might better find out for himself. Numerous illustrations are given to show the conditions of the experiments and the apparatus required for them is simple and of a kind readily procured at small expense—mostly such as the ingenous pupil will take much interest in making for himself. The introduction of this work can not fail to produce excellent results.
B. F. F.

Annals of Mathematics. Edited by Ormond Stone: W. M. Thornton, R. S. Woodward, James McMahon, W. H. Echols, Associate Editors. Office of Publication: University of Virginia. $2.00 per year in advance.

The July number contains the following articles: Note on the Theory of Functions, by W. H. Echols; Homogenious Strains, by W. H. Metzler.

It also contains the solutions to ten problems.

The Mathematical Magazine: A Journal of Elementary and Higher Mathematics. Edited and published by Artemas Martin, A. M., Ph. D., LL. D. Large quarto. Issued quarterly. Price, $2.00 per year in advance.

Besides the valuable Papers and solutions, the October No., 1893, contains two pages of editorial items in which are biographies of Dr. Joel E. Henricks and Ruben Davis, Sr., and five pages of Notices of Books and Periodicals.